New birational invariants

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Birational Geometry

Rationality

An algebraic variety X is *rational* over $\mathbb{C}$ if $\mathbb{C}(X) \cong \mathbb{C}(x_1, \ldots, x_n)$, where $n = \dim_{\mathbb{C}} X$.

Example

Let $X \subset \mathbb{P}^2$ be a smooth cubic curve.

- Hodge numbers: $h^{1,0}(X) = 1$.
- Since $h^{1,0}(X) \neq 0$, X is **not rational**.

Example (Surfaces (dim 2))

A smooth cubic $X \subset \mathbb{P}^3$ is rational.

Intermediate Jacobian and Cubic Threefolds

Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold.

Intermediate Jacobian (Clemens-Griffiths)

For X with $\dim_{\mathbb{C}} X = 3$:

$$J^2(X) := \frac{(H^{2,1})^*}{H^3(X, \mathbb{Z})}$$

Key properties:

- Principally polarized abelian variety of dimension $\frac{1}{2} b_3(X) = 5$
- Obstruction to rationality: $J^2(X)$ is **not** isomorphic to $\mathrm{Jac}(C)$ for any curve C

Example

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 1 & & 0 & & 1 & \\ 0 & & 5 & & 5 & & 0 \\ & 1 & & 0 & & 1 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$

For a smooth cubic threefold, key Hodge numbers are $h^{\mathbf{1},\mathbf{1}}(X) = 1$ and $h^{\mathbf{2},\mathbf{1}}(X) = 5$.

Clemens-Griffiths Irrationality Criterion

Theorem (Clemens-Griffiths (1972))

*A smooth cubic threefold $X \subset \mathbb{P}^4$ is irrational because its intermediate Jacobian $J^2(X)$ is the Jacobian of a non-singular curve **only if** X is birational to $\mathbb{P}^3$.*

Key Computation

For $X \subset \mathbb{P}^4$:

$$H^{2,1}(X) \cong \mathbb{C}^5, \quad H^{1,2}(X) \cong \mathbb{C}^5, \quad H^3(X, \mathbb{Z}) \cong \mathbb{Z}^{10}$$

Thus, $J^2(X)$ is a 5-dimensional abelian variety that is not isogenous to any product of curve Jacobians.

Known Methods in Birational Geometry

- **Hodge-Theoretic Methods**
  - Intermediate Jacobian (Clemens-Griffiths).
  - Brauer group (Artin-Mumford).
- **Geometric Methods**
  - Birational automorphisms (Iskovskikh-Manin).
  - Degenerations (Voisin, Kollár, Pirutka).
- **Analytic Methods**
  - Multiplier ideal sheaves (Nadel, Ein-Lazarsfeld).

Example: Cubic Fourfold (dim 4)

Let $X \subset \mathbb{P}^5$ be a very general smooth cubic fourfold.

- Hodge diamond:

$$\begin{array}{ccc} & 1 & \\ & 1 & \\ 1 & 21 & 1 \\ & 1 & \\ & 1 & \end{array}$$

- Is X rational? Katzarkov-Kontsevich-Pantev-Yu: **no**.

Homological Mirror Symmetry

Homological Mirror Symmetry (HMS)

Statement: For a smooth projective variety X, HMS relates:

- **B-model**: Derived category $D^b_{\text{coh}}(X)$.
- **A-model**: Fukaya-Seidel category $FS(Y, W)$.

Example (Mirror of $\mathbb{P}^2$)

$$D^b(\mathbb{P}^2) \longleftrightarrow FS\left(\mathbb{C}^2, W = x + y + \frac{1}{xy}\right)$$

Hodge Structures in HMS

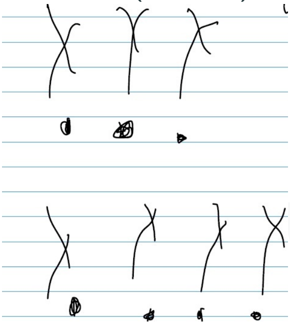
B-model (de Rham cohomology)

$$\text{for } \mathbb{P}^2: \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$\text{for the blow-up of } \mathbb{P}^2 \text{ in a point}: \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$\text{for the blow-up of } \mathbb{P}^2 \text{ in 6 points}: \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

A-model (LG mirror)



ncHodge structure[1] on periodic cyclic homology links A/B-models.

[1] Non-commutative Hodge structure encoding GW invariants.

Monodromy Operator and Rationality

Let $X \to Z_t$ be a family of smooth threefolds.

Monodromy Action

The monodromy operator acts on $H^2(Z_t, \mathbb{Z})$:

$$\mu : H^2(Z_t, \mathbb{Z}) \to H^2(Z_t, \mathbb{Z}), \quad \mu = \mathrm{diag}(1, \epsilon, \epsilon^2), \ \epsilon^3 = 1.$$

Example (Cubic Threefold)

For $X \subset \mathbb{P}^4$, μ is non-nilpotent $\implies$ X is irrational (Clemens-Griffiths).

Theorem (Katzarkov-Przyjalkowski)

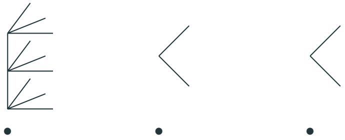
*Let X be a smooth Fano threefold with $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $X \not\cong \mathbb{P}^3$. Then X is rational **if and only if** the monodromy operator μ is nilpotent.*

Three-Dimensional Cubic: LG Mirror

The LG mirror of a cubic threefold $X \subset \mathbb{P}^4$ is the fibration by open K3 surfaces given by the potential :

$$W = \frac{(x+y+z)^3}{xyz} + z \quad \text{on } \mathbb{C}^3.$$

This family of K3 surfaces has three singular fibers - two fibers with ordinary double points and one open-book singularity.



Let $\mathcal{F}$ be the perverse sheaf of vanishing cycles of the potential. Then,

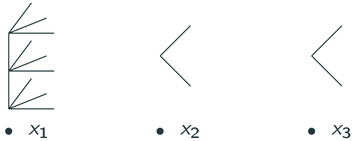
$$\dim \mathbb{H}^1(\mathcal{F}) = 5, \quad \dim \mathbb{H}^2(\mathcal{F}) = 4, \quad \dim \mathbb{H}^3(\mathcal{F}) = 5.$$

A+B

H_{dR}+Eingevalues of Quantum Multiplication

Splitting the Hodge Structure

Suppose we further split the cohomology of X into generalized eigenspaces for the operator K of quantum multiplication by $c_1(X)$, or equivalently, split the cohomology of $\mathcal{F}$ according to the critical values of the potential. In that case, we obtain as a piece a Hodge structure $\mathcal{H}$ which is exactly the Clemens-Griffiths invariant:

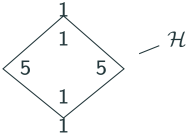


critical values of LG model

$= $ eigenvalues of K: x_1, x_2, x_3

Splitting of $\mathrm{H}^*(X) = \mathcal{H} + (1) + (1)$

$\Longrightarrow$



Theory of Atoms: Key Equation

Quantum differential equation:

$$\left(\frac{\partial}{\partial u} - \frac{1}{u^2}\mathsf{K} + \frac{1}{u}\mathsf{G}\right)\psi(u) = 0$$

- K: Quantum multiplication by $c_1(X)$.
- G: Connection matrix (flat coordinates).

Birational Invariants via Quantum Decomposition

Theorem (Katzarkov-Kontsevich-Pantev-Yu)

For a projective variety X:

- ***Decomposition***: $\mathrm{H}^*(X)$ *splits into* H_{λ_i}, *labeled by eigenvalues of* K.
- ***Birational invariance***: *Elementary pieces* H_{λ_i} *(modulo codimension* ≥ 2*) are birational invariants.*

Applications

- Singular fibers of LG mirror $\leftrightarrow$ eigenvalues of K.
- Integral Hodge structure on $\mathbb{H}^i(\mathcal{F})$ is computable via $\mathrm{QH}(X)$.

Atoms and Euler Fields

Hodge Subspace and Euler Field

- Let X be a complex projective variety. Consider the subspace of even Hodge classes:

$$H_{\text{Hodge}}(X) := \bigoplus_i \left(H^{i,i}(X) \cap H^{2i}(X, \mathbb{Q}) \right)$$

- This defines a purely even Frobenius manifold $\mathcal{F}_X$ over $\overline{\mathbb{Q}}(y)$.
- The *Euler field* $Eu \in \Gamma(\mathcal{F}_X, T_{\mathcal{F}_X})$ is:

$$Eu = c_1(T_X) + \sum_{\substack{i \\ \deg \Delta_i \neq 2}} \frac{\deg \Delta_i - 2}{2} t_i \Delta_i$$

- At a generic $p \in \mathcal{F}_X$, the spectrum of $Eu \star \cdot$ gives a μ-fold spectral cover.

Definition (Atoms)

Atoms_X are the connected components of this spectral cover.

- *Key Example*: If K_X is nef, Atoms_X has only one element (quantum product increases filtration).

- Consider the set:

$$\bigsqcup_{\text{iso classes of } X/\mathbb{C}} \text{Atoms}_X/\text{Aut}(X)$$

- By Iritani's theorem (arXiv:2307.13555), blowups induce equivalence relations:

$$\mathcal{F}_{\mathrm{Bl}_Z X} \sim \mathcal{F}_X \times \mathcal{F}_Z^{\times(m-1)} \quad (\text{codim } Z = m)$$

- Non-rationality criterion: If X has an atom not appearing in varieties of dimension $\leq \dim X - 2$, X is irrational.

To distinguish atoms, we associate invariants with them:

- **Rank ρ_α:** The rank of $\mathrm{H}_{\text{Hodge}}(X) \otimes \overline{\mathbb{Q}}(\mathbf{y})$ in the α-eigenspace.
- **Hodge polynomial $P_\alpha(t) \in \mathbb{Z}[t, t^{-1}]$:** The coefficients are given by:

$$\text{Coeff}_{t^k} P_\alpha(t) = \dim \bigoplus_{p-q=k} \mathrm{H}^{p,q}(X)_\alpha$$

Some atoms

Known Atoms from Dimensions ≤ 2

- For **points and curves**, the t^2 coefficient of $P_\alpha(t)$ is 0.
- The same holds for most **surfaces**, with the notable exceptions being K3 surfaces and surfaces of general type.
- If X is a minimal resolution of an ADE singularity from a K3 or general type surface, then $K_X \geq 0$, which implies the rank $\rho_\alpha \geq 3$.

Example: The Cubic Fourfold (I)

Hodge Numbers and a Special Point

For a very general cubic fourfold $X \subset \mathbb{CP}^5$, the key non-trivial Hodge numbers are:

$$h^{3,1}(X) = h^{1,3}(X) = 1, \quad h^{2,2}(X) = 21$$

We consider a special point $\gamma \in \mathcal{F}_X$ where the eigenvalues of the operator $\mathrm{Eu}_\gamma \star_\gamma$ are found to be:

$$\{0,\ 9,\ 9e^{2\pi i/3},\ 9e^{4\pi i/3}\}$$

The Eigenspace for Eigenvalue 0

The generalized eigenspace V_0 corresponding to the eigenvalue 0 has dimension 24 and contains the transcendental part of the cohomology.

Its Hodge structure and rank are:

- $\dim(V_0 \cap \mathrm{H}^{p-q=\pm 2}) = 1$
- $\dim(V_0 \cap \mathrm{H}^{p-q=0}) = 22$
- The rank is $\rho_0 = 2$.

The other three eigenspaces are 1-dimensional.

Example: The Cubic Fourfold (II)

Isolating an Exotic Atom

As we deform from the special point γ to a generic point in the moduli space, the large eigenspace V_0 may split further.

This process isolates at least one Hodge atom, α, with the following crucial properties derived from the structure of V_0:

- **Rank:** $\rho_\alpha \leq 2$
- **Hodge Polynomial:** $\mathrm{Coeff}_{t^2}(P_\alpha) = \dim(V_0 \cap \mathrm{H}^{p-q=2}) = 1$

Conclusion: Non-Rationality

An atom with $\mathrm{Coeff}_{t^2}(P_\alpha) = 1$ cannot come from any variety of dimension ≤ 2 (e.g., points, curves, or non-K3/general type surfaces).

Since the cubic fourfold ($N = 4$) possesses an atom that cannot appear in dimensions $\leq N - 2 = 2$, we conclude that **a very general cubic fourfold is not rational**.

Asymptotics of the Quantum Differential Equation

Quantum differential equation (QDE):

$$\left(\frac{\partial}{\partial u} - \frac{\mathsf{K}}{u^2} + \frac{\mathsf{G}}{u}\right)\psi = 0$$

Eigenvalues: Asymptotic solutions $\psi(u) \sim e^{\sigma/u}$ correspond to eigenvalues σ of $\mathsf{K} \star \cdot$.

Theorem (Non-rationality criterion)

Let X be a Fano hypersurface of degree d in $\mathbb{P}^{N-1}$. Define:

$$\delta := \dim X - 2 \cdot \frac{N-d}{d}.$$

If $\delta > \dim X - 2$, then X is not rational.

Example (4D Quartic)

$X \subset \mathbb{P}^5$, $d = 4$, $N = 6$:

$$\delta = 4 - 2 \cdot \frac{6-4}{4} = 3 > 2 \quad (\dim X - 2 = 2) \implies \text{not rational.}$$

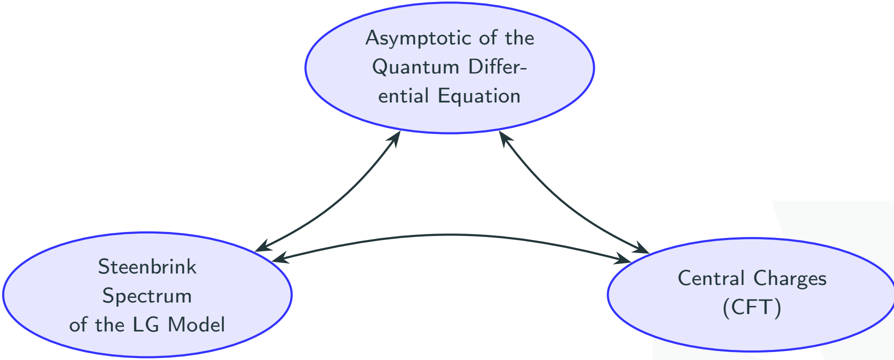
Example (5D quartic)

$$\delta = 5 - 2\left(\frac{7-4}{4}\right) = 5 - 3\frac{1}{2} > 3$$

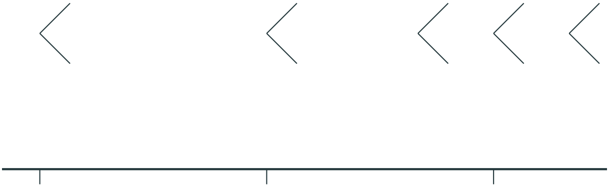
Example (3D cubic)

For the three-dimensional generic cubic:

$$\delta = 3 - 2\left(\frac{5-3}{3}\right) = \frac{5}{3} > 3 - 2 \implies \text{not rational.}$$


Asymptotic of the Quantum Differ- ential Equation
Steenbrink Spectrum of the LG Model
Central Charges (CFT)

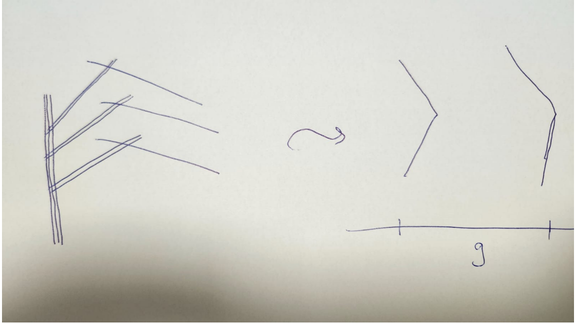
X_k a 2-dimensional cubic with $\mathrm{Pic}(X_k) \cong \mathbb{Z}_2$.



$\mathbb{P}^1$ $\quad$ $\#\mathrm{sing} = 8 - \deg = 5$

$$|2K_{\mathbb{P}^1} + 5| \neq 0$$

On the LG side: $H^0 + H^2 + H^4 = \mathbb{Z}^9$, Steenbrink=0.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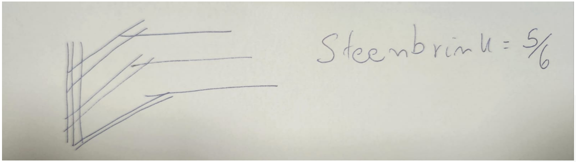
Steenbrink = 5/6

G-equivariant atoms, with L. Cavenaghi and M. Kontsevich

Setup:

- Let $X_{\text{geom}} \subset (\mathbb{P}^1)^4$ be a smooth hypersurface of degree $(1,1,1,1)$ over an algebraically closed field k.
- **Key Fact**: X_{geom} is the blowup of $(\mathbb{P}^1)^3$ at an elliptic curve E.

Atomic Structure

- 8 simple "point-like" atoms.
- 1 atom α_E linked to E.

New Setup:

- Define X over a non-closed field k.
- Assume the Galois group mixes the 4 factors of $(\mathbb{P}^1)^4$.

Key Calculation

At the "naive" point $q_i = 1, t_j = 0$:

$$\text{Eigenvalues of } Eu \star \cdot \;=\left\{ \underbrace{\lambda_1}_{\text{mult } 1}, \underbrace{\lambda_2}_{\text{mult } 4}, \underbrace{\lambda_3}_{\text{mult } 7} \right\}$$

- The third piece has Hodge numbers: 5 (middle), 1 (top/bottom).
- Only 2 algebraic cycles defined over k.

Thank you!